$\boxed{\text{Day 6 - AM}}$

a set of vectors $\{v_1, v_2, \dots, v_p\}$ is <u>linearly independent</u> if the equation

$c_1v_1 + c_2v_2 + \dots + c_pv_p = 0$ has only the trivial solution where all $c_i$'s $= 0$.

(no vector $v_i$ can be written as a linear combination of the other vectors.)

if $H$ is a subspace of $V$, a set of vectors $B = \{b_1, b_2, \dots b_p\}$ in $V$ is a <u>basis</u> for $H$ if

1) $B$ is a linearly independent set
2) $H = \text{Span}\{b_1, \dots, b_p\}$

<u>Note:</u> $V$ is a subspace of itself, thus you can also find a basis for the entire vector space.

<u>Ex</u>: let $W = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} : a - 3b - c = 0 \right\}$ be a subset of $\mathbb{R}^3$.

$a = 3b + c$ then $W = \left\{ \begin{bmatrix} 3b+c \\ b \\ c \end{bmatrix} \right\}$

$$\begin{bmatrix} 3b+c \\ b \\ c \end{bmatrix} = b \underbrace{\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}}_{v_1} + c \underbrace{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}_{v_2}$$ so $W = \text{Span}\{v_1, v_2\}$

*Thm $\rightarrow$ Since $v_1, v_2 \in \mathbb{R}^3$, we can say $\text{Span}\{v_1, v_2\} = W$ is a subspace of $\mathbb{R}^3$.

then a basis for $W$ would be $\left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

Thus, Spans are subspaces and every subspace is a Span of its basis.

Ex: basis for $\mathbb{R}^n = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \right\}$

basis for $M_{2\times 2}(\mathbb{R}) = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

Ex: Consider $\left\{ \underset{a_1}{\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}}, \underset{a_2}{\begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix}}, \underset{a_3}{\begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}}, \underset{a_4}{\begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}} \right\}$

Are they independent? Row Reduce!

$$\begin{bmatrix} 1 & 0 & 2 & 0 \\ -2 & 3 & -1 & 0 \\ 3 & -1 & 5 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

No! $a_3 = 2a_1 + a_2$

So $\{a_1, a_2, a_3, a_4\}$ are linearly dependent but $\{a_1, a_2, a_4\}$ is a basis for $\text{Col}[a_1\ a_2\ a_3\ a_4] = \mathbb{R}^3$

Thm: If a vector space $V$ has a basis $B = \{b_1, \ldots, b_n\}$ then any set in $V$ containing more than $n$ vectors must be linearly dependent

Thm: If a vector space $V$ has a basis of $n$ vectors, then every basis of $V$ must contain exactly $n$ vectors.

$\longrightarrow$

# Eigenvectors and Eigenvalues

An eigenvector of an $n \times n$ matrix $A$ is a nonzero vector $x$ such that $Ax = \lambda x$ for some scalar $\lambda$. Then $\lambda$ is called the eigenvalue.

Ex: Consider $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$ and $u = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$

Is $u$ an Eigenvector of $A$?

$$Au = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ -5 \end{bmatrix} = \begin{bmatrix} -24 \\ 20 \end{bmatrix} = -4 \begin{bmatrix} 6 \\ -5 \end{bmatrix} = -4u$$

Yes! $u$ is an eigenvector of $A$ corresponding to an eigenvalue of $-4$.